

A note on static metrics: the degenerate case

Joan Josep Ferrando¹ and Juan Antonio Sáez²

¹ Departament d'Astronomia i Astrofísica, Universitat de València, E-46100 Burjassot, València, Spain.

² Departament de Matemàtiques per a l'Economia i l'Empresa, Universitat de València, E-46071 València, Spain

E-mail: joan.ferrando@uv.es; juan.a.saez@uv.es

Abstract. We give the necessary and sufficient conditions for a 3-metric to be the adapted spatial metric of a static vacuum solution. This work accomplishes for the degenerate cases the already known study for the regular ones (*Class. Quantum Grav.* **23** (2006) 569-571).

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For a static metric $g = -e^{2U(x^k)}dT^2 + \gamma_{ij}(x^k)dx^i dx^j$, the vacuum field Einstein equations reduce to the following coupled system involving the 3-dimensional *spatial metric* γ and the *potential function* U :

$$\text{tr } R = 0, \quad R = \nabla\chi + \chi \otimes \chi, \quad \chi = dU. \quad (1)$$

where $R \equiv R(\gamma)$ denotes the Ricci tensor of the metric γ and ∇ its covariant derivative.

In [1] the following question was established: what are the necessary and sufficient conditions for a 3-metric to be the spatial metric of a static vacuum solution? In other words, what minimal conditions must γ satisfy to guarantee that a U exists such that the pair $\{\gamma, U\}$ is a solution of (1)? This problem was solved in [1] for the family of spatial metrics with a non-degenerate Ricci tensor. Here we complete this study by solving the problem for metrics with a degenerate Ricci tensor.

A static metric g has real Weyl eigenvalues, i.e., the magnetic part of the Weyl tensor vanishes. Moreover the electric Weyl tensor $E \equiv E(g)$ is the Ricci tensor of the spatial metric γ , $E = R$. Consequently:

Lemma 1 *The spatial metric γ has a degenerate Ricci tensor if, and only if, the space-time metric g is of Petrov-Bel type D.*

Of course, a degenerate Ricci tensor means a double eigenvalue. We do not consider the case of a triple eigenvalue which imposes g to be the flat metric. On the other hand, the non degenerate case corresponds to Petrov-Bel type *I* space-times.

The γ -Ricci identities for the vector $\chi = dU$ lead to [1]:

$$C_{ij} = \eta_j^{pq}(2R_{ip}\gamma_{qk} + \gamma_{ip}R_{qk})\chi^k, \quad (2)$$

where C is the Cotton-York tensor which, under the traceless Ricci condition, is

$$C_{ij} = -\eta_j^{pq} \nabla_q R_{ip}. \quad (3)$$

When the Ricci tensor is not degenerate there are three independent equations in the five equations (2). Then, the vector χ can be obtained in terms of metric concomitants and one arrives to the Bartnik and Tod result [1]. Here, we state it as follows:

Theorem 1 (Bartnik and Tod) *Let γ be a 3-dimensional metric, $R \equiv R(\gamma)$ its Ricci tensor and $s \equiv \text{tr } R^2$, $t \equiv \text{tr } R^3$. The necessary and sufficient conditions for γ to be the spatial metric of a type I static vacuum solution are:*

$$\text{tr } R = 0, \quad \Delta \equiv 12t^2 - 2s^3 \neq 0, \quad (4)$$

$$R = \nabla \chi + \chi \otimes \chi, \quad \chi \equiv \frac{1}{\Delta} (6R^2 - s\gamma)^2 (*[R, C]) \quad (5)$$

$C \equiv C(\gamma)$ being the Cotton-York tensor of γ .

For a vacuum type D metric the Ricci tensor of the spatial metric γ has a simple eigenvector e and an eigenplane with projector σ . If μ is the associated double eigenvalue, we have:

$$R = -2\mu e \otimes e + \mu\sigma = \mu(-3e \otimes e + \gamma). \quad (6)$$

If we put expression (6) on the right-hand side of equation (2) we obtain,

$$C(e, e) = 0, \quad C_\perp = 0, \quad (7)$$

where \perp denotes the e -orthogonal projection, $(C_\perp)_{ij} = \sigma_i^k \sigma_j^l C_{kl}$. Consequently, there are two independent equations in (2) and we can not obtain χ from it. In fact, the only equations that (2) imposes on χ can be written as:

$$C(e) = 3\mu * (e \wedge \chi). \quad (8)$$

On the other hand, from equations (7) and the Bianchi identities, $\nabla \cdot R = 0$, we obtain:

$$\nabla e = e \otimes dm - e(m)\gamma, \quad m \equiv \ln |\mu|^{\frac{1}{3}}. \quad (9)$$

where $e(m) = \gamma(e, dm) = e^k \partial_k m$. The expression above implies that all the metric concomitants can be written in terms of e , and μ and its derivatives.

Now if we compute the Cotton-York tensor by using (9), we can solve equation (8) and obtain χ up to an undetermined function λ :

Lemma 2 *For a type D static vacuum solution, the gradient of the potential, $\chi = dU$, takes the expression:*

$$\chi = -dm + \lambda e, \quad m \equiv \ln |\mu|^{\frac{1}{3}}, \quad (10)$$

where e and μ are, respectively, the simple eigenvector and the double eigenvalue of the Ricci tensor of the spatial metric γ .

In order to determine the function λ , we put expression (10) in the static vacuum equations (1) and, if we make use of (9), we obtain:

$$R = -\nabla dm + dm \otimes dm + [e(\lambda) + \lambda^2]e \otimes e - \lambda e(m)\gamma. \quad (11)$$

On the other hand, from the Ricci identities for the vector e and taking into account (9) and (11), we have:

$$\lambda e(m)e = d[e(m)]. \quad (12)$$

When $e(m) \neq 0$ the above equation enables us to determine λe and from Lemma 2 we obtain an expression of χ .

Remark 1 All the type D static vacuum metrics were obtained by Ehlers and Kundt [2] whom distinguished three invariant classes: A , B and C -metrics. The B -metrics are those with a gradient of the Weyl eigenvalue laying on the space-like principal two-plane (see also [3], where the intrinsic characterization of these three classes and of the Schwarzschild solution was accomplished). Then, in terms of the spatial metric γ a B -metric is characterized by $e(m) = 0$.

Consequently, so far our study has enabled us to obtain χ for the A -metrics and C -metrics. In order to give its explicit expression note that the modulus of the double eigenvalue may be computed from $6\mu^2 = \text{tr } R^2$, and the projector on the simple direction may be acquired from the equality $18\mu^2 e \otimes e = 6R^2 - s\gamma$. Moreover, (10) and (12) show that actually we can obtain the potential U or $V = e^U$ in terms of the Ricci tensor. After all these considerations we can state.

Theorem 2 *Let γ be a 3-dimensional metric, $R \equiv R(\gamma)$ its Ricci tensor and $s \equiv \text{tr } R^2$, $t \equiv \text{tr } R^3$. The necessary and sufficient conditions for γ to be the spatial metric of a A -metric or a C -metric are:*

$$\text{tr } R = 0, \quad 6t^2 = s^3, \quad P(ds) \neq 0, \quad P \equiv 6R^2 - s\gamma, \quad (13)$$

$$R = \frac{1}{V} \nabla dV, \quad V \equiv [P(ds^{-2/3}, ds^{-2/3})]^{1/2}. \quad (14)$$

The A -metrics are the vacuum type D solutions with a gradient of the Weyl eigenvalue laying on the time-like principal plane [3]. Then, we can distinguish the A -metrics (respectively, the C -metrics) by adding condition $P(ds) \wedge ds = 0$ (respectively, $P(ds) \wedge ds \neq 0$) to the conditions of theorem 2.

When $e(m) = 0$ (B -metrics), making use of (6) and (9), the ee -component of equation (11) leads to:

$$e(\lambda) + \lambda^2 = -2\mu - (dm)^2. \quad (15)$$

This last equation in λ always admits solution. For every solution λ the vector χ given in (10) satisfies the static vacuum equations (1) which can now be written substituting expression (15) in equation (11). Then we obtain a condition solely involving γ -concomitants which can be written in terms of the Ricci tensor R .

Theorem 3 *Let γ be a 3-dimensional metric, $R \equiv R(\gamma)$ its Ricci tensor and $s \equiv \text{tr } R^2$, $t \equiv \text{tr } R^3$. The necessary and sufficient conditions for γ to be the spatial metric of a B -metric are:*

$$\text{tr } R = 0, \quad 6t^2 = s^3, \quad P(ds) = 0, \quad P \equiv 6R^2 - s\gamma, \quad (16)$$

$$R = \Omega^{-\frac{1}{3}} \nabla d\Omega^{\frac{1}{3}} - 2[\Omega + (d\Omega)^2]P, \quad \Omega \equiv -\frac{s}{18t}. \quad (17)$$

Remark 2 Once we have characterized the spatial metric γ of a static vacuum solution we can look for the potential U which completes the space-time metric g . Note that the potential is defined up to an additive constant because only its gradient appears in vacuum equations (1). One can take up this constant by changing the static time t with a constant factor. Then, we can refer to the family of potentials that differ by an additive constant as unique potential.

Theorems 1 and 2 show that for type I static vacuum solutions and for the A and C -metrics the potential U is unique. Nevertheless, in the case of the B -metrics the gradient of the potential depends on λ , a solution of (15). From (9) a function β exists such that $e = \mu^{-\frac{1}{3}} d\beta$. Moreover, $d(\lambda e) = 0$ and, consequently, $\lambda = \mu^{\frac{1}{3}} f(\beta)$. Then, (15) takes the expression:

$$f' + f^2 = K, \quad K \equiv -\mu^{-1/3}[2\mu + (dm)^2]. \quad (18)$$

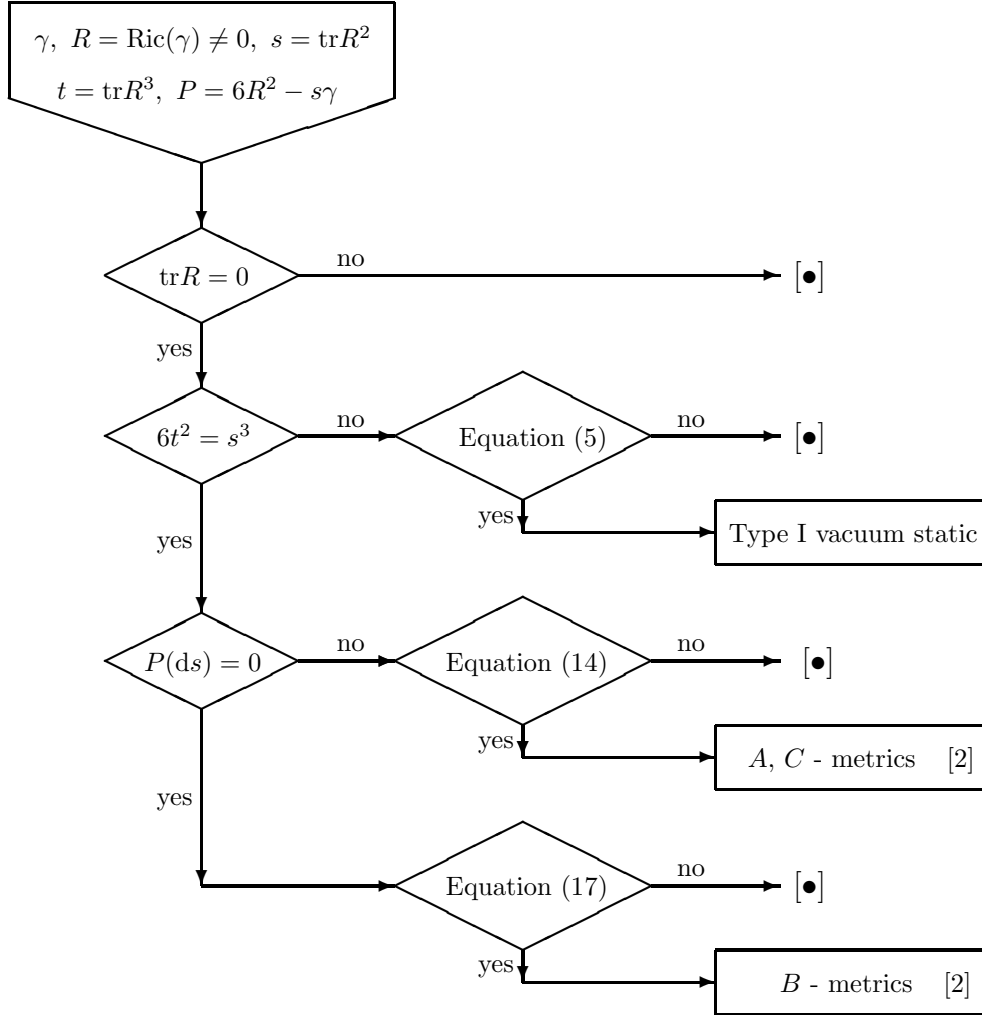
From (16) and (17) we can show that the metric concomitant K is constant. Moreover it coincides with the curvature of the two-dimensional metric of the time-like principal plane of a B -metric [3]. For a B_3 -metric one has $K = 0$ and equation (18) admits solution $f = 0$ and a one-parametric family of solutions. For a B_1 -metric ($K > 0$) or a B_2 -metric ($K < 0$) equation (18) admits a one-parametric family of solutions. In any case, one can obtain a canonical form for the metric γ and one shows that the associated space-time metric g is independent of the potential U obtained with the different solutions λ .

The non uniqueness of the potential U for a given spatial metric γ was underlined by Tod [4]. He found a family of 'spatial metrics which are static in many ways', and the resulting space-time metrics g are always of Petrov type D . He also claims that all the g associated with a γ are diffeomorphic as we have stated above. Our approach presented here also shows that the spatial metrics studied by Tod are precisely those generating the B -metrics.

Remark 3 In this paper we have accomplished the intrinsic characterization of the 3-metrics γ which are the spatial metric of a static vacuum solution g . A different problem, albeit with a similar statement is to intrinsically characterize the space-time metric g itself. We have solved this problem for type D static vacuum solutions [3], and in [5] we have given an intrinsic characterization of the type I static metrics.

Theorems 1, 2 and 3 can be summarized in the following algorithm presented below as a flow chart. Horizontal arrows pointing the symbol $[\bullet]$ indicate that γ is not the

spatial metric of a static vacuum solution.



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